ON NORMAL FORMS OF SINGULAR LEVI-FLAT REAL ANALYTIC HYPERSURFACES

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ABSTRACT. Let $F(z) = \mathcal{R}e(P(z)) + h.o.t$ be such that $M = (F = 0)$ defines a germ of real analytic Levi-flat at $0 \in \mathbb{C}^n$, $n \geq 2$, where $P(z)$ is a homogeneous polynomial of degree k with an isolated singularity at $0 \in \mathbb{C}^n$ and Milnor number μ. We prove that there exists a holomorphic change of coordinate ϕ such that $\phi(M) = (\mathcal{R}e(h) = 0)$, where $h(z)$ is a polynomial of degree $\mu + 1$ and $j_0^k(h) = P$.

1. Introduction and Statement of the results

Let M be a germ at $0 \in \mathbb{C}^n$ of a real codimension one irreducible analytic set. For the sake of simplicity we will denote germs and representative of germs by the same letter. Since M is real analytic of codimension one and irreducible, it can be defined in $\mathbb{C}^n$ by $(F = 0)$, where F is an irreducible germ of real analytic function. The singular set of M is defined by $sing(M) = (F = 0) \cap (dF = 0)$ and its smooth part $(F = 0)\backslash(dF = 0)$ will denoted by M^*. The Levi distribution L on M^* is defined by $L_p := ker(\partial F(p)) \subset T_pM^* = ker(dF(p))$, for any $p \in M^*$.

Definition 1.1. We say M is Levi-flat if the Levi distribution on M^* is integrable.

Remark 1.2. The integrability condition of L implies that M^* is tangent to a real codimension one foliation $\mathcal{L}$. Since the hyperplanes L_p, $p \in M^*$, are complex, the leaves of $\mathcal{L}$ are complex codimension one holomorphic submanifolds immersed on M^*.

Remark 1.3. If the hypersurface M is defined by $(F = 0)$ then the Levi distribution L on M can be defined by the real analytic 1-form $\eta = i(\partial F - \bar{\partial} F)$, which will be called the Levi 1-form of F. The integrability condition is equivalent to $(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F|_{M^*} = 0$

In the case of a real analytic smooth Levi-flat hypersurface M in $\mathbb{C}^n$, its local structure is very well understood, according to E. Cartan, around each $p \in M$ we can find local holomorphic coordinates $z_1, \ldots, z_n$ such that $M = \{\mathcal{R}e(z_1) = 0\}$.

More recently D. Burns and X. Gong [B-G] have proved an analogous result in the case $M = F^{-1}(0)$ Levi-flat, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$, $n \geq 2$, is a germ of real analytic function such that

$$F(z_1, \ldots, z_n) = \mathcal{R}e(z_1^2 + \ldots + z_n^2) + h.o.t.$$

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They show that there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\phi(M) = (\mathcal{R}e(z_1^2 + \ldots + z_n^2) = 0)$.

In [C-L], the authors prove the above result by using the theory of holomorphic foliations. In this paper we are interested in finding similar normal forms in a situation more general. Our main result is the following:

Theorem 1. *Let $M = F^{-1}(0)$, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$, $n \geq 2$, be a germ of irreducible real analytic function such that*

(1) *$F(z_1, \ldots, z_n) = \mathcal{R}e(P(z_1, \ldots, z_n)) + h.o.t$, where P is a homogeneous polynomial of degree k with an isolated singularity at $0 \in \mathbb{C}^n$.*
(2) *The Milnor number of P at $0 \in \mathbb{C}^n$ is μ.*
(3) *M is Levi-flat.*

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\phi(M) = (\mathcal{R}e(h) = 0)$, where $h(z)$ is a polynomial of degree $\mu + 1$ and $j_0^k(h) = P$.

Remark 1.4. In particular, we obtain the result of [B-G].

Theorem 2. *In the same spirit we have the following generalization: Let $M = F^{-1}(0)$, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$, $n \geq 3$, be a germ of irreducible real analytic function such that*

(1) *$F(z_1, \ldots, z_n) = \mathcal{R}e(Q(z_1, \ldots, z_n)) + h.o.t$, where Q is a quasihomogeneous polynomial of degree d with an isolated singularity at $0 \in \mathbb{C}^n$.*
(2) *M is Levi-flat.*

Then there exists a germ of biholomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that

$$\phi(M) = (\mathcal{R}e(Q(z) + \sum_j c_j e_j(z)) = 0),$$

where $e_1, \ldots, e_s$ are the elements of the monomial basis of the local algebra of Q such that $deg(e_j) > d$ and $c_j \in \mathbb{C}$.

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2. Preliminaries

Let us fix some notations that will be used from now on.

(1) $\mathcal{O}_n$: The ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$.
$\mathcal{O}(U)$ = set of holomorphic functions in the open set $U \subset \mathbb{C}^n$.
(2) $\mathcal{O}_n^* = \{f \in \mathcal{O}_n / f(0) \neq 0\}$.
$\mathcal{O}^*(U) = \{f \in \mathcal{O}(U) / f(z) \neq 0, \forall z \in U\}$.
(3) $\mathcal{M}_n = \{f \in \mathcal{O}_n / f(0) = 0\}$ maximal ideal of $\mathcal{O}_n$.
(4) $\mathcal{A}_n$: The ring of germs at $0 \in \mathbb{C}^n$ of complex valued real analytic functions.
(5) $\mathcal{A}_{n\mathbb{R}}$: The ring of germs at $0 \in \mathbb{C}^n$ of real valued real analytic functions. Note that $F \in \mathcal{A}_n$ is in $\mathcal{A}_{n\mathbb{R}}$ if and only if $F = \bar{F}$.
(6) $Diff(\mathbb{C}^n, 0)$: The group of germs at $0 \in \mathbb{C}^n$ of holomorphic diffeomorphisms $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with the operation of composition.
(7) $j_0^k(f)$: The k-jet at $0 \in \mathbb{C}^n$ of $f \in \mathcal{O}_n$.

Definition 2.1. Two germs $f, g \in \mathcal{O}_n$ are said to be right equivalent, if there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $f \circ \phi^{-1} = g$.

The local algebra of $f \in \mathcal{O}_n$ is by definition

$$A_f = \mathcal{O}_n/(\partial f/\partial z_1, \ldots, \partial f/\partial z_n).$$

Definition 2.2. Define by $\mu(f,0) := dim A_f$, the Milnor number of f at $0 \in \mathbb{C}^n$.

Morse Lemma can now be rephrased by saying that if $0 \in \mathbb{C}^n$ is an isolated singularity of f with Milnor number $\mu(f,0) = 1$ then f is right equivalent to its second jet. The next lemma is a generalization of Morse's Lemma. We refer to [A-G-V], pg.121.

Lemma 2.3. *Suppose $0 \in \mathbb{C}^n$ is an isolated singularity of $f \in \mathcal{M}_n$ with Milnor number μ. Then f is right equivalent to $j_0^{\mu+1}(f)$.*

2.1. **The complexification.** In this section we state some general facts about complexification of germs of real analytic functions.

Given $F \in \mathcal{A}_n$; we can write its Taylor series at $0 \in \mathbb{C}^n$ as

$$F(z) = \sum_{\mu,\nu} F_{\mu\nu} z^\mu \bar{z}^\nu, \tag{2.1}$$

where $F_{\mu\nu} \in \mathbb{C}$, $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$, $z^\mu = z_1^{\mu_1} \ldots z_n^{\mu_n}$, $\bar{z}^\nu = \bar{z}_1^{\nu_1} \ldots \bar{z}_n^{\nu_n}$. When $F \in \mathcal{A}_{n\mathbb{R}}$ then the coefficients $F_{\mu\nu}$ satisfy

$$\bar{F}_{\mu\nu} = F_{\nu\mu}.$$

The complexification $F_{\mathbb{C}} \in \mathcal{O}_{2n}$ of F is defined by the series

$$F_{\mathbb{C}}(z,w) = \sum_{\mu,\nu} F_{\mu\nu} z^\mu w^\nu. \tag{2.2}$$

If $F \in \mathcal{A}_{n\mathbb{R}}$, $F(0) = 0$ and $M = F^{-1}(0)$ defines a Levi-flat, the complexification $\eta_{\mathbb{C}}$ of its Levi 1-form $\eta = i(\partial F - \bar{\partial} F)$ can be written as

$$\eta_{\mathbb{C}} = i(\partial_z F_{\mathbb{C}} - \partial_w F_{\mathbb{C}}) = i \sum_{\mu,\nu} (F_{\mu\nu} w^\nu d(z^\mu) - F_{\mu\nu} z^\mu d(w^\nu)).$$

The complexification $M_{\mathbb{C}}$ of M is defined as $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0)$ and its smooth part is $M_{\mathbb{C}}^* = M_{\mathbb{C}} \backslash (dF_{\mathbb{C}} = 0)$. The integrability condition of $\eta = i(\partial F - \bar{\partial} F)|_{M^*}$ implies that $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*}$ is integrable. Therefore $\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*} = 0$ defines a foliation $\mathcal{L}_{\mathbb{C}}$ on $M_{\mathbb{C}}^*$ that will be called the complexification of $\mathcal{L}$.

Definition 2.4. The algebraic dimension of $sing(M)$ is the complex dimension of the singular set of $M_{\mathbb{C}}$.

Consider a germ at $0 \in \mathbb{C}^2$ of real analytic Levi-flat $M = (F = 0)$, where F is irreducible in $\mathcal{A}_{2\mathbb{R}}$. Let $F_{\mathbb{C}}$, $M_{\mathbb{C}} = (F_{\mathbb{C}} = 0) \subset (\mathbb{C}^4, 0)$ and $M_{\mathbb{C}}^*$ be as before. We will assume that the power series that defines $F_{\mathbb{C}}$ converges in a neighborhood of $\bar{\triangle} = \{(z,w) \in \mathbb{C}^4 / |z|, |w| \leq 1\}$, so that $F_{\mathbb{C}}(z, \bar{z}) = F(z)$ for all $|z| \leq 1$.

Let $V := M_{\mathbb{C}}^* \backslash sing(\eta_{\mathbb{C}}|_{M_{\mathbb{C}}^*})$ and denote L_p the leaf of $\mathcal{L}_{\mathbb{C}}$ through p, where $p \in V$. In this situation we have the following important Lemma of [C-L].

Lemma 2.5. *In the above situation, for any $p = (z_0, w_0) \in V$ the leaf L_p is closed in $M^*_{\mathbb{C}}$.*

In the proof of theorem 1 we will use the following result of [C-L].

Theorem 2.6. *Let $M = F^{-1}(0)$ be a germ of an irreducible real analytic Levi-flat hypersurface at $0 \in \mathbb{C}^n$, $n \geq 2$, with Levi 1-form $\eta = i(\partial F - \bar{\partial} F)$. Assume that the algebraic dimension of $sing(M) \leq 2n-4$. Then there exists an unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, if one of the following conditions is fulfilled:*

1. *$n \geq 3$ and $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 3$.*
2. *$n \geq 2$, $cod_{M^*_{\mathbb{C}}}(sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})) \geq 2$ and $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.*

Moreover, in both cases the foliation $\mathcal{F}_M$ has a non-constant holomorphic first integral f such that $M = (Re(f) = 0)$.

3. Proof of theorem 1

Let $M = F^{-1}(0) \subset (\mathbb{C}^n, 0)$ be a Levi-flat, where $F(z) = \mathcal{R}e(P(z)) + h.o.t$ with P be a homogeneous polynomial of degree $k \geq 2$ with an isolated singularity at $0 \in \mathbb{C}^n$ and Milnor number μ. We want to prove that there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $\phi(M) = (\mathcal{R}e(h) = 0)$, where h is a polynomial of degree $\mu + 1$.

The idea is to use theorem 2.6 to prove that there exists a germ $f \in \mathcal{O}_n$ such that the foliation $\mathcal{F}$ defined by $df = 0$ is tangent to M and $M = (\mathcal{R}e(f) = 0)$. The foliation $\mathcal{F}$ can viewed as an extension to a neighborhood of $0 \in \mathbb{C}^n$ of the Levi foliation $\mathcal{L}$ on M^*.

Suppose for a moment that $M = (\mathcal{R}e(f) = 0)$ and let us conclude the proof. Without lost of generality, we can suppose that f is not a power in $\mathcal{O}_n$. In this case $\mathcal{R}e(f)$ is irreducible (cf. [C-L]). This implies that $\mathcal{R}e(f) = U.F$, where $U \in \mathcal{A}_{n\mathbb{R}}$ and $U(0) \neq 0$. Let $\sum_{j \geq k} f_j$ be the taylor series of f, where f_j is a homogeneous polynomial of degree j, $j \geq k$. Then

$$\mathcal{R}e(f_k) = j^k_0(\mathcal{R}e(f)) = j^k_0(U.F) = U(0).\mathcal{R}e(P(z_1, \ldots, z_n)).$$

Hence $f_k(z_1, \ldots, z_n) = U(0).P(z_1, \ldots, z_n)$. We can suppose that $U(0) = 1$, so that

$$f(z) = P(z) + h.o.t \tag{3.1}$$

In particular, $\mu = \mu(f, 0) = \mu(P, 0)$, $f \in \mathcal{M}_n$, because P has isolated singularity at $0 \in \mathbb{C}^n$. Hence by lemma 2.3, f is right equivalent to $j^{\mu+1}_0(f)$, i.e. there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $h := f \circ \phi^{-1} = j^{\mu+1}_0(f)$. Therefore, $\phi(M) = (\mathcal{R}e(h) = 0)$ and this will conclude the proof of theorem 1.

Let us prove that we can apply theorem 2.6. We can write

$$F(z) = \mathcal{R}e(P(z_1, \ldots, z_n)) + H(z_1, \ldots, z_n),$$

where $H : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$ is a germ of real-analytic function and $j^k_0(H) = 0$. For simplicity, we assume that P has real coefficients. Then we get the complexification

$$F_{\mathbb{C}}(z, w) = \frac{1}{2}(P(z) + P(w)) + H_{\mathbb{C}}(z, w)$$

and $M_{\mathbb{C}} = F_{\mathbb{C}}^{-1}(0) \subset (\mathbb{C}^{2n}, 0)$. In the general case, replacing $P(w) = \sum a_j w^j$ by $\tilde{P}(w) = \sum \bar{a}_j w^j$, we will recover each step of proof.

Since $P(z)$ has an isolated singularity at $0 \in \mathbb{C}^n$, we get $sing(M_{\mathbb{C}}) = \{0\}$, and so the algebraic dimension of $sing(M)$ is 0. On other hand, the complexification of $\eta = i(\partial F - \bar{\partial} F)$ is

$$\eta_{\mathbb{C}} = i(\partial_z F_{\mathbb{C}} - \partial_w F_{\mathbb{C}}).$$

Recall that $\eta|_{M^*}$ and $\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}$ and $\mathcal{L}_{\mathbb{C}}$. Now we compute $sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$. We can write $dF_{\mathbb{C}} = \alpha + \beta$, with

$$\alpha = \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial z_j} dz_j := \frac{1}{2} \sum_{j=1}^{n} (\frac{\partial P}{\partial z_j}(z) + A_j) dz_j$$

and

$$\beta = \sum_{j=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial w_j} dw_j := \frac{1}{2} \sum_{j=1}^{n} (\frac{\partial P}{\partial w_j}(w) + B_j) dw_j,$$

where $\frac{1}{2}\sum_{j=1}^{n} A_j dz_j = \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial z_j} dz_j$ and $\frac{1}{2}\sum_{j=1}^{n} B_j dw_j = \sum_{j=1}^{n} \frac{\partial H_{\mathbb{C}}}{\partial w_j} dw_j$.

Then $\eta_{\mathbb{C}} = i(\alpha - \beta)$, and so

$$\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}} = (\eta_{\mathbb{C}} + i dF_{\mathbb{C}})|_{M^*_{\mathbb{C}}} = 2i\alpha|_{M^*_{\mathbb{C}}} = -2i\beta|_{M^*_{\mathbb{C}}}. \tag{3.2}$$

In particular, $\alpha|_{M^*_{\mathbb{C}}}$ and $\beta|_{M^*_{\mathbb{C}}}$ define $\mathcal{L}_{\mathbb{C}}$. Therefore $sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}})$ can be splited in two parts. Let $M_1 = \{(z,w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial w_j} \neq 0 \text{ for some } j = 1, \ldots, n\}$ and $M_2 = \{(z,w) \in M_{\mathbb{C}} | \frac{\partial F_{\mathbb{C}}}{\partial z_j} \neq 0 \text{ for some } j = 1, \ldots, n\}$, note that $M_{\mathbb{C}} = M_1 \cup M_2$; if we denote by

$$X_1 := M_1 \cap \{\frac{\partial P}{\partial z_1}(z) + A_1 = \ldots = \frac{\partial P}{\partial z_n}(z) + A_n = 0\}$$

and

$$X_2 := M_2 \cap \{\frac{\partial P}{\partial w_1}(w) + B_1 = \ldots = \frac{\partial P}{\partial w_n}(w) + B_n = 0\},$$

then $sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = X_1 \cup X_2$. Since $P \in \mathbb{C}[z_1, \ldots, z_n]$ has an isolated singularity at $0 \in \mathbb{C}^n$, we conclude that $cod_{M^*_{\mathbb{C}}} sing(\eta_{\mathbb{C}}|_{M^*_{\mathbb{C}}}) = n$.

If $n \geq 3$, we can directly apply Theorem 2.6 and the proof ends. In the case $n = 2$, we are going to prove that $\mathcal{L}_{\mathbb{C}}$ has a non-constant holomorphic first integral.

We begin by a blow-up at $0 \in \mathbb{C}^4$. Let $F(x,y) = \mathcal{R}e(P(x,y)) + h.o.t$ and $M = F^{-1}(0)$ Levi-flat. Its complexification can be written as

$$F_{\mathbb{C}}(x,y,z,w) = \frac{1}{2} P(x,y) + \frac{1}{2} P(z,w) + H_{\mathbb{C}}(x,y,z,w).$$

We take the exceptional divisor $D = \mathbb{P}^3$ of the blow-up $\pi : (\tilde{\mathbb{C}}^4, \mathbb{P}^3) \to (\mathbb{C}^4, 0)$ with homogeneous coordinates $[a : b : c : d]$, $(a,b,c,d) \in \mathbb{C}^4 \backslash \{0\}$. The intersection of the strict transform $\tilde{M}_{\mathbb{C}}$ of $M_{\mathbb{C}}$ by π with the divisor $D = \mathbb{P}^3$ is the surface

$$Q = \{[a : b : c : d] \in \mathbb{P}^3 / P(a,b) + P(c,d) = 0\},$$

which is an irreducible smooth surface.

Consider for instance the chart $(W, (t,u,z,v))$ of $\tilde{\mathbb{C}}^4$ where

$$\pi(t,u,z,v) = (t.z, u.z, z, v.z) = (x,y,z,w).$$

We have

$$F_{\mathbb{C}} \circ \pi(t,u,z,v) = z^k(\frac{1}{2}P(t,u) + \frac{1}{2}P(1,v) + zH_1(t,u,z,v)),$$

where $H_1(t,u,z,v) = H(tz,uz,z,vz)/z^{k+1}$, which implies that

$$\tilde{M}_{\mathbb{C}} \cap W = (\frac{1}{2}P(t,u) + \frac{1}{2}P(1,v) + zH_1(t,u,z,v) = 0)$$

and so $Q \cap W = (z = P(t,u) + P(1,v) = 0)$.

On the other hand, as we have seen in (3.2), the foliation $\mathcal{L}_{\mathbb{C}}$ is defined by $\alpha|_{M^*_{\mathbb{C}}} = 0$, where

$$\alpha = \frac{1}{2}\frac{\partial P}{\partial x}dx + \frac{1}{2}\frac{\partial P}{\partial y}dy + \frac{\partial H_{\mathbb{C}}}{\partial x}dx + \frac{\partial H_{\mathbb{C}}}{\partial y}dy.$$

In particular, we get

$$\pi^*(\alpha) = z^{k-1}(\frac{1}{2}\frac{\partial P}{\partial x}(t,u)zdt + \frac{1}{2}\frac{\partial P}{\partial y}(t,u)zdu + \frac{1}{2}kP(t,u)dz + z\theta),$$

where $\theta = \pi^*(\frac{\partial H_{\mathbb{C}}}{\partial x}dx + \frac{\partial H_{\mathbb{C}}}{\partial y}dy)/z^k$.

Hence, $\tilde{\mathcal{L}}_{\mathbb{C}}$ is defined by

$$\alpha_1 = \frac{1}{2}\frac{\partial P}{\partial x}(t,u)zdt + \frac{1}{2}\frac{\partial P}{\partial y}(t,u)zdu + \frac{1}{2}kP(t,u)dz + z\theta. \tag{3.3}$$

Since $Q \cap W = (z = P(t,u) + P(1,v) = 0)$, we see that Q is $\tilde{\mathcal{L}}_{\mathbb{C}}$-invariant. In particular, $S := Q \backslash sing(\tilde{\mathcal{L}}_{\mathbb{C}})$ is a leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$. Fix $p_0 \in S$ and a transverse section $\sum$ through p_0. Let $G \subset Diff(\sum, p_0)$ be the holonomy group of the leaf S of $\tilde{\mathcal{L}}_{\mathbb{C}}$. Since $dim(\sum) = 1$, we can think that $G \subset Diff(\mathbb{C}, 0)$. Let us prove that G is finite and linearizable.

At this part we use that the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ are closed (see lemma 2.5).

Let $G' = \{f'(0)/f \in G\}$ and consider the homomorphism $\phi : G \to G'$ defined by $\phi(f) = f'(0)$. We assert that ϕ is injective. In fact, assume that $\phi(f) = 1$ and by contradiction that $f \neq id$. In this case $f(z) = z + a.z^{r+1} + \ldots$, where $a \neq 0$. According to [L], the pseudo-orbits of this transformation accumulate at $0 \in (\sum, 0)$, contradicting that the leaves of $\tilde{\mathcal{L}}_{\mathbb{C}}$ are closed. Now, it suffices to prove that any element $g \in G$ has finite order (cf. [M-M]). In fact, if $\phi(g) = g'(0)$ is a root of unity then g has finite order because ϕ is injective. On the other hand, if $g'(0)$ was not a root of unity then g would have pseudo-orbits accumulating at $0 \in (\sum, 0)$ (cf. [L]). Hence, all transformations of G have finite order and G is linearizable.

This implies that there is a coordinate system w on $(\sum, 0)$ such that $G = \langle w \to \lambda w \rangle$, where λ is a d^{th}-primitive root of unity (cf. [M-M]). In particular, $\psi(w) = w^d$ is a first integral of G, that is $\psi \circ g = \psi$ for any $g \in G$.

Let Z be the union of the separatrices of $\mathcal{L}_{\mathbb{C}}$ through $0 \in \mathbb{C}^4$ and $\tilde{Z}$ be its strict transform under π. The first integral ψ can be extended to a first integral $\varphi : \tilde{M}_{\mathbb{C}} \backslash \tilde{Z} \to \mathbb{C}$ be setting

$$\varphi(p) = \psi(\tilde{L}_p \cap \sum),$$

where $\tilde{L}_p$ denotes the leaf of $\tilde{\mathcal{L}}_{\mathbb{C}}$ through p. Since ψ is bounded (in a compact neighborhood of $0 \in \sum$), so is φ. It follows from Riemann extension theorem

that φ can be extended holomorphically to $\tilde{Z}$ with $\varphi(\tilde{Z}) = 0$. This provides the first integral and finishes the proof of theorem 1.

4. Quasihomogeneous polynomials

In this section, we state some general facts about normal forms of quasihomogeneous polynomials.

Definition 4.1. The Newton support of germ $f = \sum a_{ij} x^i y^j$ is defined as $supp(f) = \{(i,j) : a_{ij} \neq 0\}$.

Definition 4.2. A holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is said to be quasihomogeneous of degree d with indices $\alpha_1, \ldots, \alpha_n$, if for any $\lambda \in \mathbb{C}$ and $(z_1, \ldots, z_n) \in \mathbb{C}^n$, we have

$$f(\lambda^{\alpha_1} z_1, \ldots, \lambda^{\alpha_n} z_n) = \lambda^d f(z_1, \ldots, z_n).$$

The index α_s is also called the weight of the variable z_s.

In the above situation, if $f = \sum a_k x^k$, $k = (k_1, \ldots, k_n)$, $x^k = x_1^{k_1} \ldots x^{k_n}$, then $supp(f) \subset \Gamma = \{k : a_1 k_1 + \ldots + a_n k_n = d\}$. The set Γ is called the diagonal. Usually one takes $\alpha_i \in \mathbb{Q}$ and $d = 1$.

One can define the quasihomogeneous filtration of the ring $\mathcal{O}_n$. It consists of the decreasing family of ideals $\mathcal{A}_d \subset \mathcal{O}_n$, $\mathcal{A}_{d'} \subset \mathcal{A}_d$ for $d < d'$. Here $\mathcal{A}_d = \{Q :$ degrees of monomials from $supp(Q)$ are $deg(Q) \geq d\}$; (the degree is quasihomogeneous).

When $\alpha_1 = \ldots = \alpha_n = 1$, this filtration coincides with the usual filtration by the usual degree.

Definition 4.3. A function f is called semiquasihomogeneous if $f = Q + F'$, where Q is quasihomogeneous of degree d of finite multiplicity and $F' \in \mathcal{A}_{d'}$, $d' > d$.

We will use the following result (cf. [A]).

Theorem 4.4. *Let f be a semiquasihomogeneous function, $f = Q + F'$ with quasihomogeneous Q of finite multiplicity. Then f is right equivalent to the function $Q + \sum_j c_j e_j(z)$, where $e_1, \ldots, e_s$ are the elements of the monomial basis of the local algebra A_Q such that $deg(e_j) > d$ and $c_j \in \mathbb{C}$.*

Example 4.5. If $f = Q + F'$ is semiquasihomogeneous and $Q(x,y) = x^2 y + y^k$, then f is right equivalent to Q. Indeed, the base of the local algebra $\mathcal{O}_2/(xy, x^2 + k y^{k-1})$ is $1, x, y, y^2, \ldots, y^{k-1}$ and lies below the diagonal Γ. Here $\mu(Q, 0) = k + 1$.

5. Proof of theorem 2

Let $M = F^{-1}(0)$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$ of real analytic Levi-flat hypersurface, where $F(z) = \mathcal{R}e(Q(z)) + h.o.t$ and Q is a quasihomogeneous polynomial of degree d with an isolated singularity at $0 \in \mathbb{C}^n$. It is easily seen that $sing(M_{\mathbb{C}}) = \{0\}$ and $cod_{M^*_{\mathbb{C}}} sing(\mathcal{L}_{\mathbb{C}}) \geq 3$. The argument is essentially the same of the proof of theorem 1. In this way, there exists an unique germ at $0 \in \mathbb{C}^n$ of holomorphic codimension one foliation $\mathcal{F}_M$ tangent to M, moreover $\mathcal{F}_M$: $dh = 0$, $h(z) = Q(z) + h.o.t$ and $M = (\mathcal{R}e(h) = 0)$. Acoording

to theorem 4.4, there exists $\phi \in Diff(\mathbb{C}^n, 0)$ such that $h \circ \phi^{-1}(w) = Q(w) + \sum_k c_k e_k(w)$, where c_k and e_k as above. Hence

$$\phi(M) = (\mathcal{R}e(Q(w) + \sum_k c_k e_k(w)) = 0).$$

6. Applications

Here we give some applications of theorem 1.

Example 6.1. $Q(x,y) = x^2y + y^3$ is a homogeneous polynomial of degree 3 with an isolated singularity at $0 \in \mathbb{C}^2$ and Milnor number $\mu(Q,0) = 4$. According to [A-G-V] pg. 184, any germ $f(x,y) = x^2y + y^3 + h.o.t$ is right equivalent to $x^2y + y^3$.

In particular, if $F(z) = \mathcal{R}e(x^2y + y^3) + h.o.t$ and $M = (F = 0)$ is a germ of real analytic Levi-flat at $0 \in \mathbb{C}^2$, theorem 1 implies that there exists a holomorphic change of coordinate such that

$$M = (\mathcal{R}e(x^2y + y^3) = 0).$$

Example 6.2. If $Q(x,y) = x^5 + y^5$ then $f(x,y) = Q(x,y) + h.o.t$ is right equivalent to $x^5 + y^5 + c.x^3y^3$, where $c \neq 0$ is a constant (cf. [A-G-V], pg. 194). Let $F(z) = \mathcal{R}e(x^5 + y^5) + h.o.t$ be such that $M = (F = 0)$ is Levi-flat, theorem 1 implies that there exists a holomorphic change of coordinate such that

$$M = (\mathcal{R}e(x^5 + y^5 + c.x^3y^3) = 0).$$

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